Maximal subsemigroups of finite semigroups

Monoids of partial order-endomorphisms

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Wilf Wilson

University of St Andrews

Why compute maximal subsemigroups?

- It seemed like a nice problem!

Our starting point. . .

N. Graham, R. Graham, J. Rhodes
Maximal subsemigroups of finite semigroups
Journal of Combinatorial Theory, 4:203–209, 1968.

Our ending points. . .

C. Donoven, J. D. Mitchell, W. A. Wilson
Computing maximal subsemigroups of a finite semigroup preprint arXiv:1606.05583, 2016.

J. East, J. Kumar, J. D. Mitchell, W. A. Wilson
Maximal subsemigroups of finite transformation and diagram monoids preprint arXiv:1706.04967, 2017.

Brief definition of Green's relations

On a semigroup S, define the equivalences:

- $x \mathscr{L} y \Leftrightarrow x$ and y generate the same left ideal.
- $x \mathscr{R} y \Leftrightarrow x$ and y generate the same right ideal.
- $x \mathscr{J} y \Leftrightarrow x$ and y generate same two-sided ideal.

A $\mathscr{J}$-class is an equivalence class of $\mathscr{J}$, and so on.

The main ideas behind our algorithm

From *Graham, Graham, Rhodes*:

- A maximal subsemigroup M contains all but one $\mathscr{J}$-class, J.
- $M \cap J$ has one of a few forms.

First step: algorithm for Rees 0-matrix semigroups.

Second step: algorithm for arbitrary semigroups.

- Process each $\mathscr{J}$-class that contains a generator.
- *(Perform computations on the principal factor.)*
- Construct some graphs and digraphs from the $\mathscr{R}$-classes and $\mathscr{L}$-classes of the $\mathscr{J}$-class.
- Analyse graphs to find maximal subsemigroups.

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Example: order-endomorphisms, etc

i.e. monoids of order-preserving & -reversing partial transformations.

For a partial transformation α and $i, j \in \operatorname{dom}(\alpha)$:

- order-preserving: $i \leqslant j \Rightarrow i\alpha \leqslant j\alpha$;
- order-reversing: $i \leqslant j \Rightarrow i\alpha \geqslant j\alpha$.

The maximal subsemigroups of some, but not all, of these related monoids have been described.

Partial order-endomorphisms

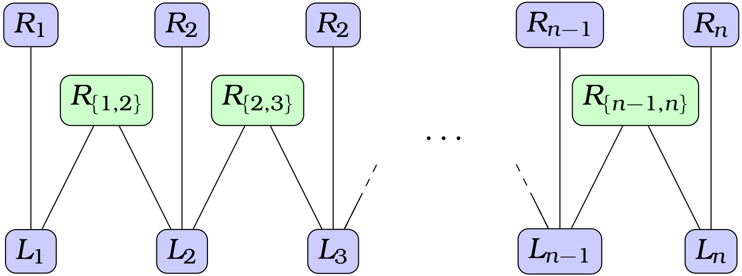

Figure: The graph for $\mathcal{PO}_n$.

Maximal (independent subset → subsemigroup)

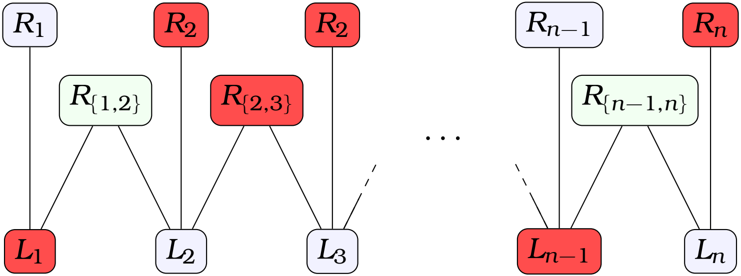

Figure: The graph for $\mathcal{PO}_n$.

Partial order-endomorphisms

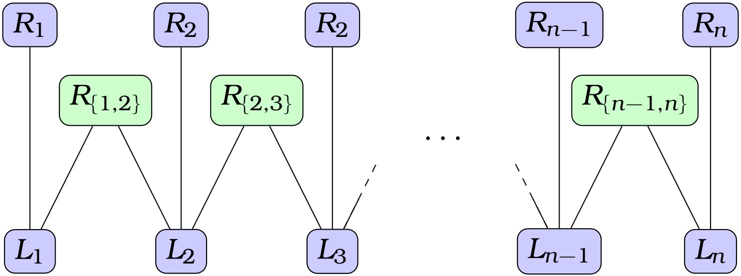

Figure: The graph for $\mathcal{PO}_n$.

Total order-endomorphisms

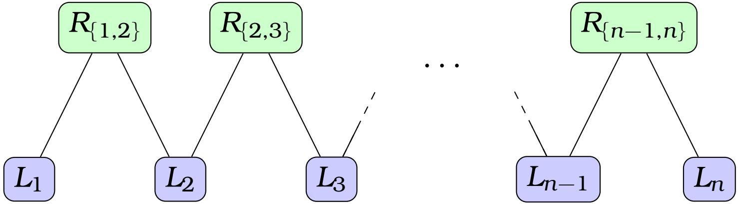

Figure: The graph for $\mathcal{O}_n$.

Partial injective order-endomorphisms

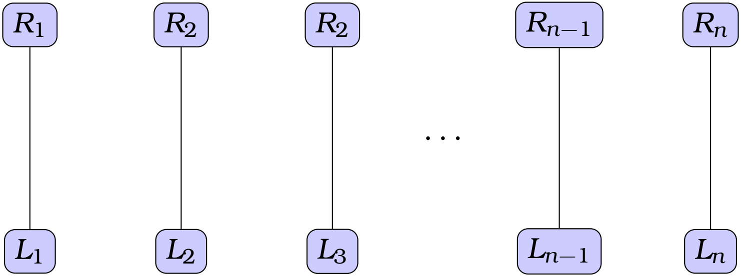

Figure: The graph for $\mathcal{POI}_n$.

Partial order-(anti)endomorphisms

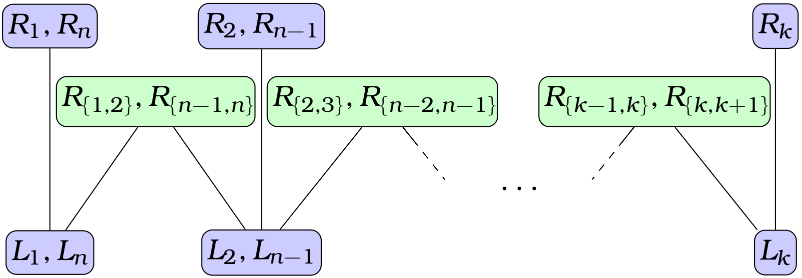

Figure: The graph for $\mathcal{POD}_n$, when $n = 2k - 1$ is odd.

Total order-(anti)endomorphisms

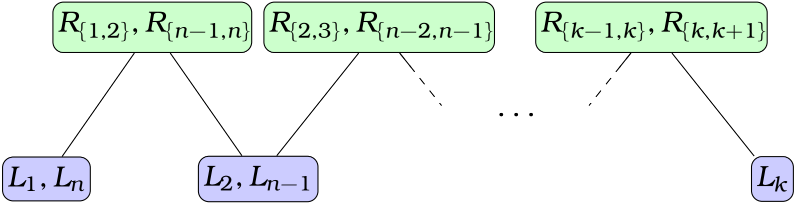

Figure: The graph for $\mathcal{OD}_n$, when $n = 2k - 1$ is odd.

Partial injective order-(anti)endomorphisms

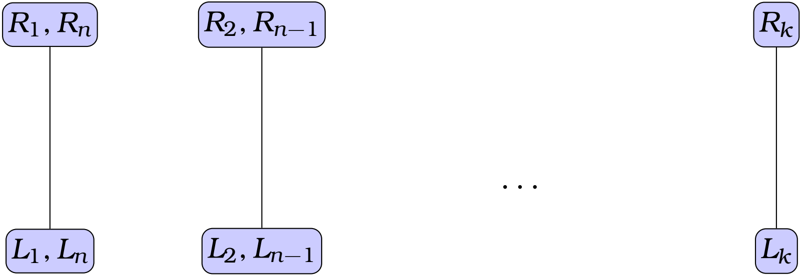

Figure: The graph for $\mathcal{PODI}_n$, when $n = 2k - 1$ is odd.

Computing with SEMIGROUPS in GAP

```
gap> S := PartialOrderEndomorphisms(10);;
gap> Size(S);
4780008
gap> NrMaximalSubsemigroups(S);
1042
gap> T := OrderAntiEndomorphisms(8);;
gap> MaximalSubsemigroups(T);
[ <transformation monoid of degree 8 with 10 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
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  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators>,
  <transformation monoid of degree 8 with 55 generators> ]
```

http://gap-packages.github.io/Semigroups

Semigroups

Download .tar.gz

View On GitHub

Version 3.0.4

This project is maintained by
J. D. Mitchell

GAP Package Semigroups

The Semigroups package is a GAP package containing methods for semigroups, monoids, and inverse semigroups. There are particularly efficient methods for semigroups or ideals consisting of transformations, partial permutations, bipartitions, partitioned binary relations, subsemigroups of regular Rees 0-matrix semigroups, and matrices of various semirings including boolean matrices, matrices over finite fields, and certain tropical matrices.

Semigroups contains efficient methods for creating semigroups, monoids, and inverse semigroup, calculating their Green's structure, ideals, size, elements, group of units, small generating sets, testing membership, finding the inverses of a regular element, factorizing elements over the generators, and so on. It is possible to test if a semigroup

http://gap-packages.github.io/Digraphs

Digraphs

Download .tar.gz

View On GitHub

Version 0.10.0

This project is maintained by
J. D. Mitchell

GAP Package Digraphs

The Digraphs package is a GAP package for digraphs and multidigraphs.

The current version of this package is version 0.10.0. For more information, please refer to the package manual. There is also a README file and a CHANGELOG.

Dependencies

This package requires GAP version >=4.8.2

The following other GAP packages are needed:

Thank you!